

# Mathematics Paper HCS 2003

Code—14

## MATHEMATICS

Time Allowed : 3 Hours

Maximum Marks : 150

Note : Attempt any *Five* questions. All questions carry equal marks. Q. No. 1 is compulsory. Attempt *two* questions from Part I and *two* questions from Part II. The parts of the same question must be answered together and must not be interposed between answers to other questions.

1. Attempt any *four* of the following : ( $4 \times 7\frac{1}{2} = 30$ )

(a) Prove that any homomorphism from a finite dimensional vector space onto itself is an isomorphism.

(b) Find the area bounded by the curve :

$$y = x^3 - 6x^2 + 8x$$

and  $x$ -axis.

P.T.O.

(c) The extremities of a light pole rest on two smooth pegs A and B in the same horizontal line. A heavy load hangs from the point P of the pole. If  $AP = 3 PB$  and the pressure at B is 25N more than that at A, find the weight of the load.

(d) If  $p$  and  $q$  are chosen randomly from the set  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$  with replacement, find the probability that the roots of the equation :

$$x^2 + px + q$$

are real.

(e) Two bodies of mass 9 and 16 kg are placed at a distance of 10 m on a smooth table. If they attract each other with a constant force of one N, find after what time they will meet ?

(f) Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function. Prove that there exists  $x \in [0, 1]$  such that  $f(x) = x$ .

## Part I

2. (a) Prove that if  $A$  is a  $n \times n$ -matrix with complex entries, then  $A$  can be expressed as  $\sqrt{P} + iQ$  where  $P$  and  $Q$  are Hermitian matrices. (10)

(b) Reduce the real quadratic form :

$$3x_1^2 - 3x_2^2 - 5x_3^2 - 2x_1x_2 - 6x_2x_3 - 6x_3x_1$$

to the canonical form. Find its rank and index. (20)

3. (a) (i) Define the Gamma function. Prove that if  $n$  is a positive integer, then  $\Gamma(n) = (n-1)!$ .

(ii) Find the maxima of :

$$u = x^2 + y^2 + z^2$$

$$\text{where } 2x^2 + 3y^2 + z^2 = 1.$$

(5+10)

(b) (i) Find the equation of "shortest distance between the lines :

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

$$\text{and } \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$

- (ii) Find the equation of cone with vertex at (1, 1, 1) and passing through the curve of intersection of :

$$x^2 + y^2 + z^2 = 1$$

$$\text{and } x + y + z = 1.$$

(7+8)

- 4 (a) Find the differential equation that represents all parabolas each of which has latus rectum  $4a$  and whose axes are parallel to the  $x$ -axis. (8)

- (b) Solve : (10)

$$\frac{dy}{dx} + y \tan x = y^3 \sec x.$$

- (c) Solve : (12)

$$(x^7 y^2 + 3y) dx + (3x^8 y - x) dy = 0.$$

### Part II

5. (a) Evaluate : (15)

$$\int \int_S (x^3 dy dz + x^2 y dz dx + x^2 z dy dx)$$

where  $S$  is the sphere  $x^2 + y^2 + z^2 = 1$ .

(b) (i) Prove that for vectors A and B :

$$\nabla (A \cdot B) = (A \cdot \nabla) B + (B \cdot \nabla) A + A \times (\nabla \times B) + B \times (\nabla \times A)$$

(ii) Prove that for a vector F :

$$(\nabla \cdot \nabla) F = \nabla \cdot (\nabla F) = \nabla^2 F. \quad (10+5)$$

6. (a) If  $f$  is continuous real valued function of real variable which satisfies  $f(x+y) = f(x) \cdot f(y)$  for all  $x$  and  $y$ , then prove that either  $f(x) = 0$  for all  $x$  or there is  $a > 0$  such that  $f(x) = a^x$ .

(20)

(b) Show that the series :

$$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \dots$$

converges uniformly on  $(K, +\infty)$  where  $K$  is any positive integer. (10)

7. (a) Derive the Lagrange's interpolation formula to fit the data : (10)

$x$	:	-1	1	2	3
$f(x)$	:	1	1	4	9

(b) The median and mode of the following distribution are known to be 33.50 and 34 respectively. Find the values of  $f_3, f_4, f_5$  given  $\Sigma f = 230$ . (10)

Class	$f$
0-10	4
10-20	16
20-30	$f_3$
30-40	$f_4$
40-50	$f_5$
50-60	6
60-70	4

(c) A ship is sailing westwards at 8 m/sec. While trying to fix a bolt at the top of the mast, the sailor drops the bolt. If the mast of the ship is 19.6 m high, where will the bolt hit the deck ? (Given  $g = 9.8$  m/sec.) (10)