# Mathematics Paper HCS 2004

## MATHEMATICS

*Time : 3 Hours*

*Maximum Marks* : 150

**Note :** Attempt any *Five* questions. All questions carry equal marks. Q. No. 1 is compulsory. Answer *two* questions from Part I and *two* questions from Part II. The parts of the same question must be answered together and must not be interposed between answers to other questions.



**1.** Answer any *four* of the following : **(4×7½=30)**

(a) Let $\{e_1, e_2, e_3, e_4\}$ be a basis for a vector space V over **R**. Prove that $\{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_1\}$ is also a basis of V.

(b) Let $g : \mathbf{R} \to \mathbf{R}$ be defined by :

$$g(t) = 0 \quad \text{if } t \text{ is irrational or } 0$$

$$= \frac{1}{n} \quad \text{if } t = \frac{m}{n}$$

where $m$ and $n$ are integers, $t$ is non-zero and heighest common factor of $m$ and $n$ is 1.

Prove that $g$ is continuous at all irrational $t$ and discontinuous at all rational non-zero $t$.

(c) Find the equation of the plane passing through the line :

$$\frac{x-1}{4} = \frac{y-2}{6} = \frac{z-1}{3}$$

and the point (4, 3. 7).

(d) Let ABCD be a square. Suppose forces represented in magnitude and direction by AB, 2BC, 2CD, DA and DB are acting at a point 0. Prove that they are at equilibrium.

(e) A truck is moving along a level road at the rate of 40 km/hr. In what direction a bullet must be fired from it with a velocity of 200 m/sec so that its resultant motion is perpendicular to the truck ?

(f) A random variable $x$ follows Poisson distribution such that $P(x = 1)$ is equal to $P(x = 2)$. Find $P(x > 3)$.

## Part I

(i) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear tranformation defined by :

$$T(x, y) = (2x + 3y, y + 3x)$$

Find the matrix of T with respect to the basis $\{(1, 1), (1, -1)\}$ **(10)**

(ii) Find the matrix P such that P′AP is diagonal where P′ denotes the transpose of P and A is the matrix : **(12)**

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

(iii) Let λ and μ be distinct eigen values of a Hermitian matrix H. Suppose $x$ and $y$ are eigen vectors corresponding to λ and μ respectively. Prove that $x$ and $y$ are mutually orthogonal. (8)

3. (i) Prove that if $f : [0, 1] \to \mathbf{R}$ is continuous on [0, 1] except at finitely many points, then $f$ is Riemann integrable. (10)

(ii) Find the volume of the torus generated by revolving the circle :

$$x^2 + y^2 = 4$$

about the line $x = 3$.

(iii) Determine the points where the function :

$$x^3 + y^3 - 3axy$$

has a maximum or minimum. (5)

(iv) Find the radius of curvature of the curve :

$$x^{(2/3)} + y^{(2/3)} = a^{(2/3)}$$

at the point $(a \cos^3\theta, a \sin^3\theta)$. (5)

(i) Solve the differential equation : (8)

$$y \sin 2x \, dx - (1 + y^2 + \cos x) \, dy = 0.$$

(ii) Solve :

$$(D^2 + a^2) \, y = \sin ax.$$ (12)

(iii) Solve :

$$(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0.$$ (10)

## Part II

(i) Find curl grad F, where $F = x^2y + 2xyz + z^2$. (8)

(ii) If $r$ and $a$ are two vectors, prove that curl $(r \times a) = -2a$. (7)

(iii) State Gauss's divergence theorem and use it to evaluate

$$\iint_S x^2 \, dx \, dz + y^2 \, dz \, dx + 2z(xy - x - y) dx \, dy$$

where S is the surface of the cube $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $0 \leq z \leq 1$. (15)

(i) Prove that a continuous real valued function defined on [3, 8] is uniformly continuous on [3, 8]. (10)

(ii) Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous periodic function of period 2 (*i.e.*, $f(x + 2) = f(x)$ for all $x$). Prove that $f$ is bounded and attains its bounds. (10)

(iii) Find the interval of positive reals where the series :

$$\sum_{n=0}^{\infty} x(1-x)^n$$

converges uniformly. (10)

7. (i) A stone falling from the top of a vertical tower has descended $x$ metres when another is let fall from a point $y$ metres below the top. Show that, if they fall from rest and reach the ground together, then the height of the tower is $[(x + y)^2/4x]$. (10)

(ii) Use Gauss-Jordon method to solve the system of equations :

$$x + 4y - z = -5$$
$$x + y - 6z = -12$$
$$3x - y - z = -4$$ (10)

(iii) Derive Newton's forward interpolation formula, given the values of the function $y = f(x)$ at $x_1, x_2, \ldots\ldots x_n$ and $x_k - x_{k-1} = h$ for all $k$. (10)